SEMESTRAL EXAMINATION
M. MATH I YEAR, II SEMESTER 2015-2016
COMPLEX ANALYSIS

Max. 100. Time limit: 3hrs

Notations: U is the open unit disc, $T = \partial U, Log$ is the principal logarithm.

1. If Ω is a horizontal or vertical open strip in $\mathbb{C}$ show that there is a bounded holomorphic function on Ω which is not a constant. [10]

2. Prove that there is no holomorphic function f in $\{z : |z| > 4\}$ such that $f'(z) = \frac{z^2}{(z-1)(z-2)(z-3)}$ $\forall z$.

Hint: compute the integral of the right side over a suitable closed path. [15]

3. Let f be a complex valued continuous function on $[a, b]$. If $\left|\int\limits_a^b f(t)dt\right| = \int\limits_a^b |f(t)|\, dt$ show that $e^{i\theta} f$ is non-negative for some real number θ. Use this to show that if f is holomorphic on a region Ω and $|f|$ is harmonic then f is a constant. [20]

HInt: write $\int\limits_a^b f(t)dt$ as $re^{i\theta}$ and use the fact that $\text{Re}\{e^{-i\theta} f(t)\} \leq |f(t)|$.

4. Let $f : U \to U$ be holomorphic. If f has more than one fixed point show that $f(z) = z$ $\forall z$. [15]

5. Show that the collection of all maps from $\mathbb{C}_\infty$ into itself of the type $f(z) = \frac{az+b}{cz+d}$ where a, b, c, d are integers, $ad - bc = 1$, a and d are odd, b and c are even is a group under composition of functions. [15]

6. Prove that $f(re^{i\theta}) = \sum\limits_{n \neq 0} \frac{1}{n} r^{|n|} e^{in\theta}$ defines a harmonic function on U [15]

7 Let f be holomorphic in a region Ω. Suppose that $f''(z) + f(z) = 0$ for all $z \in \Omega$ and $f(c) = f'(c) = 0$ for some $c \in \Omega$. Show that $f \equiv 0$. [10]